Chapter 4

Open-Channel Hydraulics

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1. Introduction

1.1 Laminar and turbulent flows

Pipe flow is thought to be critical when its Reynolds number reaches 2000. The same value can be applied to an open-channel flow by substituting the hydraulic radius for the characteristic length in the Reynolds number. That is, the Reynolds number in open-channel is defined as

\[ Re = \frac{VR_h}{\nu} \]  

where \( V \) = characteristic velocity, \( R_h \) = hydraulic radius and \( \nu \) = kinematic viscosity. In open-channel flows, by using \( D = 4R_h \) (here, \( D \) = pipe diameter), the flow is laminar when \( Re < 500 \), transitional when \( 500 < Re < 2000 \), and turbulent when \( Re > 2000 \). Particularly, in an open-channel, the laminar flow is hardly observed in nature unless it is made artificially (for example, a thin film of liquid flowing down an inclined or vertical plane).

1.2 Subcritical and supercritical flows

The Froude number defined below represents the relative magnitude of the inertia force to the gravity force (strictly speaking, the Froude number denotes the square root of the ratio of the inertia force to the gravity force)

\[ Fr = \frac{V}{\sqrt{gD}} \]  

where \( D \) = hydraulic depth. This dimensionless number has generally little aerodynamic...
interest, but it is of considerable importance in ship design, where gravitational (wave) force is the primary determinant of the total forces.

(Q) What is the Froude number for large slope angle $\theta$ and for $\alpha \neq 0$?

$$Fr = \frac{V}{\sqrt{gD\cos\theta/\alpha}}$$

In Eq.(2), $V$ is the flow velocity and $\sqrt{gD}$ is the celerity (wave speed) of the longwave. Under a critical condition, i.e., $Fr = 1$, the flow velocity is the same as the celerity of the long wave.

(1) $Fr < 1$: Subcritical Flow
Flow velocity is less than the celerity of the long wave. Any disturbance made at the downstream point propagates at a speed of $\sqrt{gD} - V$ in the upstream direction. So the downstream influences the upstream.

(2) $Fr > 1$: Supercritical Flow
Flow velocity is greater than the celerity of the long wave. Any disturbance made at the downstream cannot be transmitted in the upstream direction.

2. Basic Properties
2.1 Longitudinal profiles
Figure 4.1 shows a longitudinal profile of the Mississippi River, US. It can be seen that the slope changes from locations to locations. The average slope is about $310-70 \times 0.3/(485 \times 1600) = 1/10,000$. The slope is defined to be positive although the bed elevation decreases in the downslope direction. The left and right banks are referenced to a downstream-looking direction.
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Figure 4.1 Longitudinal profile of the Mississippi River, US

Figure 4.2 shows a longitudinal profile of the Matamek River, US, which is a semi-alluvial river. Interestingly, compared to the previous figure, discontinuities are seen in longitudinal profiles, bed materials, and flow conditions.

Figure 4.2 Longitudinal profile of the Matamek River, US
2.2 Cross section properties

For a cross section of an open channel, the following geometric elements are defined:

- flow depth ($y$): the vertical distance of the lowest point of a channel section from the free surface
- stage: the elevation of the free surface above a datum
- top width ($T$): the width of channel section at the free surface
- water area ($A$): the cross sectional area of the flow normal to the flow of direction
- wetted perimeter ($P$): the length of the line of intersection of the channel wetted surface with a cross sectional plane normal to the flow direction. The wetted perimeter is related to the shear stress acting in the opposite direction of the flow
- hydraulic radius ($R_h$): the ratio of the water area to its wetted perimeter
  \[ R_h = \frac{A}{P} \]
- hydraulic depth ($D$): the ratio of the water area to the top width
  \[ D = \frac{A}{T} \]
- section factor for critical flow computation ($Z$):
  \[ Z = A\sqrt{D} \]
- prismatic channel: a channel with unvarying cross-section and constant slope
### Table 4.1 Geometric functions for channel elements

<table>
<thead>
<tr>
<th>Section</th>
<th>Rectangle</th>
<th>Trapezoid</th>
<th>Triangle</th>
<th>Circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area $A$</td>
<td>$B_w y$</td>
<td>$(B_w + z y)y$</td>
<td>$z y^2$</td>
<td>$\frac{1}{8}(\theta - \sin\theta)d_o^2$</td>
</tr>
<tr>
<td>Wetted perimeter $P$</td>
<td>$B_w + 2y$</td>
<td>$B_w + 2y\sqrt{1 + z^2}$</td>
<td>$2y\sqrt{1 + z^2}$</td>
<td>$\frac{1}{2}d_o$</td>
</tr>
<tr>
<td>Hydraulic radius $R$</td>
<td>$\frac{B_w y}{B_w + 2y}$</td>
<td>$\frac{(B_w + z y)y}{B_w + 2y\sqrt{1 + z^2}}$</td>
<td>$\frac{z y}{2 \sqrt{1 + z^2}}$</td>
<td>$\frac{1}{4}\left[1 - \sin\frac{\theta}{2}\right]d_o$</td>
</tr>
<tr>
<td>Top width $B$</td>
<td>$B_w$</td>
<td>$B_w + 2z y$</td>
<td>$2z y$</td>
<td>$\left[\sin\left(\frac{\theta}{2}\right)\right]d_o$ or $\frac{2}{2\sqrt{y(d_o - y)}}$</td>
</tr>
</tbody>
</table>

\[
\frac{2dR}{3Rdy} + \frac{1}{A\ dy} = \frac{5B_w + 6y}{3y(B_w + 2y)} \quad \frac{(B_w + 2z y)(5B_w + 6y\sqrt{1 + z^2}) + 4z y^2\sqrt{1 + z^2}}{3y(B_w + z y)(B_w + 2y\sqrt{1 + z^2})} = \frac{8}{3y} \\
\frac{4(2\sin\theta + 3\theta - 5\theta\cos\theta)}{3d_o\theta(\theta - \sin\theta)\sin(\theta/2)}
\]

where $\theta = 2\cos^{-1}\left(1 - \frac{2y}{d_o}\right)$

Figure 4.3 shows a cross section of a river. The top width \( T \) is 53.0 m, and the wetted perimeter \( P \) is 53.3 m. The water area \( A \) is 45.9 m\(^2\). The hydraulic depth \( D \) (mean flow depth) is 0.87 m, and the hydraulic radius \( R_h \) is 0.86 m. The cross section of the river is a wide open channel since the width to depth ratio exceeds 10 – 15. For such wide open channels, the hydraulic radius is very close to the mean flow depth.

2.3 Velocity distributions

Most flows in the river are hydraulically-rough flows without the viscous sublayer. Then the logarithmic velocity law can be applied to those flows.

\[
\frac{\bar{u}}{u_*} = \frac{1}{\kappa} \ln \left( \frac{z}{k_s} \right) + 8.5 = \frac{1}{\kappa'} \ln \left( \frac{30 z}{k_s} \right) 
\]

(3)

where \( k_s \) is the effective roughness height. If one integrates Eq.(3) over the depth, he can obtain the relation such as
where $U$ is the depth-averaged velocity. This relation is called Keulegan’s resistance for rough flow.

An approximation to Keulegan’s relation is the Manning-Strickler power form such as

$$\frac{U}{u_*} = 8.1 \left( \frac{H}{k_s} \right)^{1/6}$$

Discussions on the figure below needed
Example 4.1 Velocity distribution

Consider measured velocity profile in Figure 4.5(a). Measured velocities at tow heights are $u_1 = 0.168 \text{ m/s} \text{ at } z_1 = 0.15 \text{ m}$ and $u_2 = 0.250 \text{ m/s} \text{ at } z_2 = 0.457 \text{ m}$. The river is 60 m wide.

(1) Find the shear velocity.

Using the relationship given by Eq.(3), we have

\[
\begin{align*}
    u_1 &= \frac{u_*}{\kappa} \ln \left( \frac{z_1}{k_s} \right) + 8.5u_* \\
    u_2 &= \frac{u_*}{\kappa} \ln \left( \frac{z_2}{k_s} \right) + 8.5u_*
\end{align*}
\]

which leads to

\[
\begin{align*}
    u_2 - u_1 &= \frac{u_*}{\kappa} \ln \left( \frac{z_2}{k_s} \right) - \frac{u_*}{\kappa} \ln \left( \frac{z_1}{k_s} \right) = \frac{u_*}{\kappa} \ln \left( \frac{z_2}{z_1} \right)
\end{align*}
\]

Therefore, we have

\[
\begin{align*}
    u_* &= \frac{\kappa(u_2 - u_1)}{\ln(z_2 / z_1)} = \frac{0.41 \times (0.25 - 0.168)}{\ln(0.457 / 0.15)} \\
    &= 0.03 \text{ m/s}
\end{align*}
\]

(2) Find the bed shear stress

The bed shear stress is given by

\[
\tau_b = \rho u_*^2 = 1,000 \times 0.03^2
\]

\[
= 0.9 \text{ Pa}
\]
2.4 Rating curve

The stage-discharge relationship is called “rating curve.” The idea of the rating curve might come from the convenience of predicting the discharge for a given stage. This relation is unique for the uniform flow. That is, the discharge is proportional to the power of 5/3 of the flow depth if the Manning’s formula is used. However, except for the uniform flow, the rating curve shows a loop. That is, during a particular flood event, the number of stages are two for a discharge. The stage is higher for the falling limb, indicating the velocity is lower in the falling stage of the flood than in the rising stage.

The rating curve might be unique for channels with fixed beds. However, for channels with mobile beds, the rating curve shifts over time because bed aggradation or degradation and changes in bedforms.
3. Uniform Flows

Depending on the temporal variability (at a particular distance), the open-channel flows are divided into steady flows and unsteady flows. Similarly, depending on the spatial variability (at a particular time), the flows can be divided into uniform flows and non-uniform flows. Thus, we have four combinations as

- steady uniform flows
- steady non-uniform flows
- unsteady uniform flows
- unsteady non-uniform flows

Since the flows which are unsteady with time cannot be uniform with distance, the unsteady uniform flows do not exist. We simply call steady uniform flows as uniform flows, steady non-uniform flows as non-uniform flows, and unsteady non-uniform flows as unsteady flows.

3.1 Characteristics of uniform flows

Uniform flows in the open channel are defined by

1. the flow depth, water area, velocity, and discharge at every section of the channel reach are constant
2. the slope of energy line ($S_e$), the slope of water surface ($S_w$), and the slope of the channel bottom ($S_o$) are the same

As stated, uniform flows are considered to be steady only because unsteady uniform flows are practically nonexistent.

A constant velocity may be interpreted as a constant mean velocity. This should mean that the
flow possesses a constant velocity at every point on the channel section within the channel reach. That is, the velocity distribution across the channel section is unaltered in the reach. Such a stable pattern is attained when the boundary layer is fully developed.

![Figure 4.6 Uniform flow in a prismatic channel](image)

Figure 4.6 Uniform flow in a prismatic channel

### 3.2 Bed shear stress of uniform flow

Uniform flow is developed if the resistance is balanced by the gravity forces. Consider a force balance in the uniform flow shown in the above figure. The gravity force acting on the fluid element is given by

$$ F_g = \gamma A dx \sin \theta $$

(6)

The bed slope is defined by  $S_0 = \tan \theta$, which is close to  $\sin \theta$  if the angle of inclination  $\theta$  is small. The gravity force is balanced by the resisting force due to shear along the wetted perimeter such as

$$ F_f = \tau_0 P dx $$

(7)

Equating Eq.(6) and Eq.(7) results in
\[ \tau_0 = \gamma R_h S_0 \]  

(8)

where \( S_0 \) denotes the channel slope (\( = \sin \theta \)). Note that \( S_0 = S_o = S_e \) for uniform flows.

### 3.3 Uniform flow formulas

#### 3.3.1 Chezy formula

In 1769, a French engineer named Antoine de Chezy proposed the following relationship for the velocity \( V \) in the open-channel:

\[ V = C \sqrt{R_h S_o} \]  

(9)

where \( C = \) Chezy coefficient, \( R_h = \) hydraulic radius, and \( S_o = \) bed slope. Eq.(9) is an empirical formula, made using data collected in canals in Paris. According to Chezy’s formula, the mean velocity is proportional to the square root of the hydraulic radius and bed slope.

In fluid mechanics, the bed shear stress is represented by

\[ \tau_0 = C_f \frac{1}{2} \rho V^2 \]  

(10)

where \( C_f \) is the flow resistance coefficient. Using Eq.(7), the resisting force by the bed shear stress can be expressed as

\[ F_f = KV^2 Pdx \]  

(11)

where \( K \) is a proportionality. Using Eqs.(6) and (11), a relationship similar to Chezy’s formula can easily be obtained with \( C = \sqrt{\gamma/K} \).
3.3.2 Manning formula

Robert Manning was an engineer for the arterial drainage and inland navigation in Office of Public Works in Ireland. In 1889, for estimating the discharge in the open channel easily, he compared seven formulas with collected data and proposed the following relationship:

\[ V = \frac{1}{n} R_h^{2/3} S_0^{1/2} \]

where \( n \) = roughness coefficient. This relationship is different from Chezy’s in that the velocity is proportional to \( R_h^{2/3} \), not to \( R_h^{1/2} \). Note that both Chezy’s and Manning’s formulas are dimensionally non-homogeneous. For this, the following remedy is used:

\[ V = \frac{C_m}{n} R_h^{2/3} S_0^{1/2} \] (12)

where the value of \( C_m \) is 1 and 1.49 for SI and English units, respectively. The following Table delivers representative Manning’s roughness coefficients for various boundary materials.

Table 4.2 Manning’s roughness coefficients

<table>
<thead>
<tr>
<th>Material</th>
<th>Typical Manning roughness coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete</td>
<td>0.012</td>
</tr>
<tr>
<td>Gravel bottom with sides</td>
<td></td>
</tr>
<tr>
<td>— concrete</td>
<td>0.020</td>
</tr>
<tr>
<td>— mortared stone</td>
<td>0.023</td>
</tr>
<tr>
<td>— riprap</td>
<td>0.033</td>
</tr>
<tr>
<td>Natural stream channels</td>
<td></td>
</tr>
<tr>
<td>Clean, straight stream</td>
<td>0.030</td>
</tr>
<tr>
<td>Clean, winding stream</td>
<td>0.040</td>
</tr>
<tr>
<td>Winding with weeds and pools</td>
<td>0.050</td>
</tr>
<tr>
<td>With heavy brush and timber</td>
<td>0.100</td>
</tr>
<tr>
<td>Flood Plains</td>
<td></td>
</tr>
<tr>
<td>Pasture</td>
<td>0.035</td>
</tr>
<tr>
<td>Field crops</td>
<td>0.040</td>
</tr>
<tr>
<td>Light brush and weeds</td>
<td>0.050</td>
</tr>
<tr>
<td>Dense brush</td>
<td>0.070</td>
</tr>
<tr>
<td>Dense trees</td>
<td>0.100</td>
</tr>
</tbody>
</table>

Source: Chow, 1959.
3.3.3 Darcy-Weisbach formula

The Darcy-Weisbach formula has a long historical background (Brown, 2003). In 1845, Weisbach proposed the following relationship for the head loss:

\[ h_L = f \frac{L V^2}{D 2g} \]

where \( f \) is the friction factor. If the above relationship is rewritten for the velocity with the use of \( R_s \), then we have

\[ V = \sqrt{\frac{8g}{f \sqrt{R_s S_0}}} \]  

(13)

At that time when Weisbach proposed the above relationship, the friction factor was not well defined. In 1857, Darcy suggested that the friction factor is related not only the wall roughness but also the pipe diameter. Later, Rouse proposed a curve for the friction factor in 1943, which was further improved by Moody in 1944. While the formula is named after two great engineers of 19-th century, many others have also aided in the effort.

In general, the friction factor \( f \) is

\[ f = f_n \left( \frac{\varepsilon}{R}, \text{Re} \right) \]  

(14)

in which \( \varepsilon \) is the roughness height. Values of roughness coefficient \( f \) are given in the Moody diagram which is obtained from experiments of pipe flows. The expressions for \( f \) are

\[ f = \begin{cases} 
\frac{24}{\text{Re}} & \text{Re} \leq 500 \\
0.223 \frac{\text{Re}^{1/4}}{\text{Re}^{1/4}} & 500 < \text{Re} \leq 25,000 
\end{cases} \]  

(15a)

(15b)

For fully-developed turbulent flows over hydraulically-smooth boundary with \( \text{Re} > 25,000 \),
\[
\frac{1}{\sqrt{f}} = 2\log_{10} \frac{Re}{k} + 0.4 \tag{16a}
\]
and for fully-developed turbulent flows over hydraulically-rough boundary with \( u^* k / \nu > 70 \) or \( \text{Re} \sqrt{f/(R k)} > 50 \),
\[
\frac{1}{\sqrt{f}} = 2\log_{10} \frac{R}{k} + 2.16 \tag{16b}
\]
where \( k \) is the equivalent size of the Nikuradse type surface roughness and \( u^* \) is the shear velocity \((= \sqrt{\tau_0 / \rho})\).

Among three resistance factors, the Darcy-Weisbach \( f \) has the best theoretical background. It is non-dimensional and its values for steady uniform flows are given in the Moody diagram. However, it should be emphasized that the roughness coefficient \( f \) in Darcy-Weisbach formula is a local quantity as indicated by the above relationship whereas the roughness coefficients of the other formulas are reach-averaged quantities. The reason why Darcy-Weisbach formula is not so popular in the practical hydraulics is that the roughness coefficient comes from the pipe flow experiments. That is, there is no Moody diagram for the open-channel flow, and \( f - \text{Re} \) relationship changes according to channel geometry.

### 3.4 Dimensional consideration

The uniform flow formulas can be expressed as
\[
S_0 = S_f \tag{17}
\]
which states that the friction slope, defined by \( S_f = \tau_0 / (\gamma R_h) \) from Eq.(8), is the same as the bed slope. In Eq.(17), the friction slope is given by
Manning’s, Chezy’s, and Darcy-Weisbach’s formulas were originally developed empirically although theoretical numerous attempts were made later. From the three relationships, it is clear that

\[ S_f = \frac{V^2}{C^2 R_h} \quad \text{if Chezy’s formula is used} \]

\[ S_f = \frac{n^2 V^2}{R_h^{4/3}} \quad \text{if Manning’s formula is used} \]

\[ S_f = \frac{fV^2}{8gR_h} \quad \text{if Darcy-Weisbach’s formula is used} \]

The above equation reveals that

(a) The Chezy \( C \) has the dimension of \( \sqrt{g} \).

(b) \( C_m \) in the Manning’s formula has the dimension of \( \sqrt{g} \) because it is unreasonable to assume \( n \) changes with changing \( g \). Therefore, the Manning’s \( n \) has the dimension of \( [L^{1/6}] \).

Although \( n \) has a dimension of \( [L^{1/6}] \), in practice the same numerical value of \( n \) is used in English system as in SI system, and hence the constant 1.49 absorbs not only the dimension of \( g \) but also the conversion factor from SI system.

**Example 4.2: Uniform flow**

Consider a steady flow of \( Q = 10 \ m^3/s \) in a 10 m wide rectangular channel. The slope of the
channel is $S_0 = 0.26/1,000$ and the friction factor is $f = 0.01$.

(1) Find the normal depth $y_n$ of the flow.

The mean velocity by Darcy-Weisbach formula is given by

$$V = \sqrt{\frac{8g}{f}} \sqrt{R_h S_0}$$

Here, $R_h \approx y$ for wide channels. Therefore, the unit discharge is given by

$$q = \sqrt{\frac{8g}{f}} y_n^{3/2} S_0^{1/2}$$

Thus, we have

$$y_n^{3/2} = q \sqrt{\frac{f}{8g}} S_0^{-1/2}$$

$$= \frac{10}{10} \times \sqrt{\frac{0.01}{8 \times 9.8}} \times \left( \frac{0.26}{1,000} \right)^{-1/2}$$

$$= 0.70$$

Therefore, the normal depth is

$$y_n = 0.79 \text{ m}$$

If we do not assume that the channel is wide, then the hydraulic radius is

$$R_h = \frac{W y_n}{W + 2y_n} = 0.68 \text{ m.}$$

which is 14% smaller than the flow depth.

(2) Find Manning’s $n$ and Chezy’s $C$.

Using the relationship given by Eq.(18), we have
\[ C = \sqrt{\frac{8g}{f}} = 88 \]

\[ n = \frac{\sqrt{f R^{1/6}}}{\sqrt{8g}} = 0.011 \]

(3) Find the bed shear stress

The bed shear stress is calculated as

\[ \tau_0 = \gamma R_s S_0 = 9,801 \times 0.681 \times 0.00026 \]

\[ = 1.74 \text{ Pa} \]

4. Energy and Momentum Principles

The energy equation derived in the fluid mechanics can hardly be applied to open-channel flows since the pressure varies vertically. Thus the energy principle, specific energy, tailored specially for open-channel flows, is used. Similarly, the specific force is the momentum principle simplified and devised only for open-channel flows.

4.1 Energy principle

4.1.1 Specific energy

Applying the energy equation at two points shown in the figure below leads to the following relationship:

\[ \frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_l \]

where
\[
\frac{p_1}{\gamma} + z_1 = y_1 + \Delta x \cdot S_0; \quad \frac{p_2}{\gamma} + z_2 = y_2
\]

If we assume that \(\alpha_1 = \alpha_2 = 1\) and \(S_0 \approx 0\), then we have

\[
\frac{V_1^2}{2g} + y_1 = \frac{V_2^2}{2g} + y_2
\]

where the head loss is also neglected. It can be seen that the sum of velocity head and water depth is constant and it is defined as the specific energy. The specific energy is the total energy per unit weight with elevation datum taken as the bottom of the channel. That is,

\[
E_s = y + \frac{V^2}{2g} = y + \frac{Q^2}{2gA^2}
\]

which is constant.

**Figure 4.7 Derivation of specific energy**

### 4.1.2 Critical depth

The specific energy has a minimum value below which the given \(Q\) cannot occur. The value of \(y\) for minimum \(E_s\) is obtained by differentiating Eq.(21), i.e.,
\[
\frac{dE_s}{dy} = 1 - \frac{Q^2}{gA^2} \frac{dA}{dy} = 1 - \frac{V^2}{gA} \frac{dA}{dy} \quad (22)
\]

Near the free surface, it holds that \( dA/dy = T \). Thus,

\[
\frac{dE_s}{dy} = 1 - \frac{V^2 T}{gA} = 1 - \frac{V^2}{gD} = 1 - Fr^2
\quad (23)
\]

where \( D \) is the hydraulic depth \((D = A/T)\). Therefore, if the specific energy is minimum for a given discharge, then it should hold that

\[
\frac{V^2}{2g} = \frac{D}{2}
\quad (24)
\]

Since no approximations about the shape of the channel are made in deriving eq.(24), it should be applied any arbitrary-shaped channel.

For rectangular channels, the critical depth is

\[
y_c = \left( \frac{q^2}{g} \right)^{1/3}\quad (25)
\]

Substituting \( hc \) into eq.(25) results the minimum value of the specific energy such as

\[
Min(E_s) = \frac{3}{2} y_c
\quad (26)
\]

4.1.3 Critical Slope

In a mild slope condition, the normal depth is higher than the critical depth. As the slope increases gradually, the normal depth is lowered with the critical depth unchanged. When two depths become identical, the slope is said to be critical. If the slope is steeper than the critical slope, it becomes steep slope.
The slope of the channel that sustains a given discharge at a uniform and critical depth is called the critical slope $S_c$. This slope can be obtained by substituting the velocity from Eq.(12) into eq.(24)

$$S_c = \frac{n^2 gD}{R_h^{4/3}}$$

(27)

Notice that the critical slope is proportional to the squared roughness. That is, the more rough channel requires the higher slope to be a critical flow for a given discharge. It is interesting to note that the uniform flow on the mild slope is a subcritical flow but the uniform flow on the steep slope is a supercritical flow.

### 4.2 Momentum principle

Consider the control volume of the open-channel flow in the figure below. The momentum equation applied to the control volume can be written as

$$\sum F = \rho Q(V_2 - V_1)$$

(28)

where the LHS of the above equation denotes the sum of the external forces acting on the control volume. That is,

$$\sum F = P_1 - P_2 + W \sin \theta - F_f$$

(29)

where $W$ is the weight of water in the control volume and $F_f$ is the friction. If we ignore both the weight of water in the control volume and friction, then we have

$$\rho Q(V_2 - V_1) = P_1 - P_2$$

(30)

or
where \( h_g \) is the vertical distance from the free surface to the centroid of \( A \). The above equation shows that the sum of two terms, defined by the specific force, is conserved at two cross sections. That is, the specific force is defined by

\[
M = \frac{Q^2}{gA} + h_g A
\]  

(32)

where the first term represents the momentum per unit time and per unit weight of water and the second term the hydrostatic force per unit weight of water.

Figure 4.8 Forces acting on the control volume of the open channel flow

If the specific force is differentiated with respect to \( h \), then

\[
\frac{dM}{dy} = -\frac{Q^2}{gA^2} \frac{dA}{dy} + \frac{d}{dy} (h_g A)
\]  

(33)

From \( dM / dy = 0 \), one has
\[ \frac{V^2}{2g} = \frac{D}{2} \]  

which indicates that the critical flow occurs if the specific force is minimum for a given discharge.

**Example 4.3: Contractions in open-channel flow**

Consider a steady flow of \( Q = 10 \ m^3/s \) in a 10 m wide rectangular channel.

From Example 4.1, the flow depth and velocity before contraction are, respectively, given by

\[ y = 0.79 \ \text{m}, \ \ V = 1.27 \ \text{m/s} \]

which enables us to estimate the specific energy such as

---

(1) Find the maximum possible elevation of a sill \( \Delta z_{\text{max}} \) that will not cause backwater.

From Example 4.1, the flow depth and velocity before contraction are, respectively, given by

\[ y = 0.79 \ \text{m}, \ \ V = 1.27 \ \text{m/s} \]

which enables us to estimate the specific energy such as
\[ E_s = y + \frac{V^2}{2g} = 0.87 \text{ m} \]

Since the unit discharge is \( q = \frac{Q}{W} = 1.0 \text{ m}^3/\text{s} \), the critical depth is
\[ y_c = \left( \frac{q^2}{g} \right)^{1/3} = 0.467 \text{ m} \]

The minimum value of the specific energy is
\[ E_{s_{\text{min}}} = \frac{3}{2} y_c = 0.70 \text{ m} \]

Since
\[ E_s = E_{s_{\text{min}}} + \Delta z_{\text{max}} \]

Therefore, the maximum elevation of the sill is given by
\[ \Delta z_{\text{max}} = 0.17 \text{ m} \]

(2) Find the maximum lateral contraction \( \Delta W_{\text{max}} \) of the channel that will not cause backwater.

Since \( E_s = \frac{3}{2} y_c \), the critical depth is given by
\[ y_c = \frac{2}{3} E_s = 0.58 \text{ m} \]

5. Rapidly Varied Flows (Hydraulic Jump)

From a practical viewpoint, hydraulic jump is a useful means of dissipating excess energy of supercritical flows. Its merit is in preventing possible erosion below overflow spillways, chutes, and sluices, for it quickly reduces the velocity of the flow on a paved apron to a point where the flow becomes incapable of scouring the downstream channel bed. The hydraulic jump used for energy dissipation is usually confined partly or entirely to a channel reach that is known as
the stilling basin. The bottom of the basin is paved to resist scouring. Hydraulic jump can also be used as mixing devices for the addition and mixing of chemicals in water and wastewater treatment plants. In natural channels, a hydraulic jump is used to provide aeration of the water for environmental considerations.

Hydraulic jump in a horizontal plane is considered herein. From the continuity relationship,

\[ V_1 y_1 = V_2 y_2 \]  

(35)

and, from the momentum equation

\[
\frac{y_1^2}{2} - \frac{y_2^2}{2} = \rho V_2 (y_2 V_2) + \rho V_2 (-y_1 V_1)
\]

(36)

Solving the above two equations together leads to

\[ y_2 = -\frac{y_1}{2} + \frac{y_1}{2} \sqrt{1 + 8 F I_1^2} \]  

(37)

or

\[ \frac{y_2}{y_1} = \frac{1}{2} \left( \sqrt{1 + 8 F I_1^2} - 1 \right) \]  

(38)

where the depths \( y_1 \) and \( y_2 \) are referred to as conjugate depths. Using eq.(38) from the information before the jump, the water depth after the jump (after the energy dissipation) can be obtained. Energy is not conserved before and after the jump. In order to obtain the energy loss, the following energy equation should be solved:

\[
\frac{V_1^2}{2g} + y_1 = \frac{V_2^2}{2g} + y_2 + h_l
\]

(39)

where \( h_l \) represents losses due to the jump. Eliminating \( V_1 \) and \( V_2 \) yields

\[ h_l = \frac{(y_2 - y_1)^3}{4 y_1 y_2} \]  

(40)

(Q) Derive the formula for the hydraulic jump in an inclined channel with a slope \( \theta \).
Example 4.4: Hydraulic jump

Consider a steady flow of Q = 10 m³/s in a 10 m wide rectangular channel. The upstream velocity \( V_1 = 4 \text{ m/s} \) is rapidly reduced to form a hydraulic jump.

(1) Find the flow depth and velocity after the jump.

Since \( V_1 = 4 \text{ m/s} \), the flow depth before the jump is \( y_1 = 0.25 \text{ m} \). Thus, Fr before the jump is given by

\[
Fr_1 = \frac{V_1}{\sqrt{gy_1}} = 2.56
\]

which indicates that the flow is supercritical. Using the formula by Eq.(38), we have

\[
y_2 = \frac{y_1}{2} \left( \sqrt{1 + 8Fr_1^2} - 1 \right) = 0.789 \text{ m}
\]

which results in \( V_2 = 1.27 \text{ m/s} \).

(2) Find the force after the jump.

The hydrostatic force after the jump can be calculated as

\[
F_2 = \frac{y_2^2}{2} \times W = 30.5 \text{ KN}
\]

Example 4.5: Flow under sluice gate

A sluice gate discharges 10 m³/s in a 10 m wide rectangular channel. The flow depth downstream of the gate is \( y_1 = 0.25 \text{ m} \) and rapidly increases to the normal depth of \( y_2 = 0.788 \text{ m} \). The hydraulic jump is located at the toe of the sluice gate.

(1) Find the water level upstream of the sluice gate.
The velocity downstream of the gate (before the jump) can be calculated as

\[ V_1 = \frac{Q}{y_1 W} = \frac{10}{0.25 \times 10} = 4 \text{ m/s} \]

Therefore, the specific energy before the jump is given by

\[ E_{y1} = y_1 + \frac{V_1^2}{2g} = 0.25 + \frac{4^2}{2g} = 1.06 \text{ m} \]

which is the same as the specific energy upstream of the sluice gate. That is,

\[ E_{y0} = y_0 + \frac{V_0^2}{2g} = 0.25 + \frac{1}{2g} \left( \frac{Q}{y_0 W} \right)^2 = 1.06 \]

which is reduced to be the third-order equations such as

\[ y_0^3 - 1.06y_0^2 + \frac{1}{2g} = 0 \]

the solution of which is

\[ y_0 = 1.0 \text{ m} \]

with \( V_0 = 1 \text{ m/s} \).

(2) What is the force acting on the sluice gate?

The momentum equation applied to the sluice gate is written as

\[ \sum F = \rho Q(V_1 - V_0) \]

The LHS of the equation is given by

\[ \sum F = \frac{y}{2} y_1^2 W - \frac{y}{2} y_0^2 W - F \]

where \( F \) is the force acting on the sluice gate (\( \rightarrow \)). Thus, we have

\[ F = \frac{y}{2} y_0^2 W - \frac{y}{2} y_1^2 W - \rho Q(V_1 - V_0) \]
how much energy is lost in the hydraulic jump?

The velocity after the jump is given by

\[ V_2 = \frac{Q}{y_2W} = \frac{10}{0.788 \times 10} = 1.27 \text{ m/s} \]

Therefore, the specific energy after the jump is given by

\[ E_{s2} = y_2 + \frac{V_2^2}{2g} = 0.788 + \frac{1.27^2}{2g} = 0.87 \text{ m} \]

Therefore, the energy lost is

\[ \Delta E = E_{s1} - E_{s2} = 1.06 - 0.87 = 0.19 \text{ m} \]

6. Gradually Varied Flows

Gradually varied flow is a steady flow whose depth changes gradually along the length of the channel. This means two assumptions are involved in the definition, namely steady flow and hydrostatic pressure distribution. The former suggests that the flow is constant with time, and the latter that the streamlines are practically parallel.

6.1 Governing Equation

Consider a control volume of water column in the next page. The total head \( H \) is

\[ H = \alpha \frac{V^2}{2g} + d \cos \theta + z \]  

(41)

where \( \alpha = \) energy correction factor, \( V = \) mean velocity, \( d = \) flow depth, and \( z = \) elevation of a
channel bottom from a certain datum. Assuming that $\alpha = 1$ and the slope is very mild, then $\cos \theta \approx 1$ and $d \approx y$. Differentiating Eq.(41) with respect to $x$ yields

$$\frac{dH}{dx} = \frac{d}{dx}\left(\frac{V^2}{2g}\right) + \frac{dy}{dx} + \frac{dz}{dx}$$

(42)

Figure 4.9 Flow in a rectangular prismatic open-channel

where

$$\frac{dH}{dx} = -S_e$$

(43a)

$$\frac{dz}{dx} = -S_0$$

(43b)

$$\frac{d}{dx}\left(\frac{V^2}{2g}\right) = -\frac{Q^2}{gA^2} \frac{dA}{dy} \frac{dy}{dx} = -\frac{Q^2T}{gA^3} \frac{dy}{dx} = -\frac{V^2}{gD} \frac{dy}{dx}$$

(43c)

It should be noted in Eq.(39) that the loss of the total head ($dH$) is always negative in the flow direction. Gradually varied flow should imply that the water depth does not change significantly in order to satisfy $dA/dy = T$. Therefore, we have
which describes the variation of the flow depth in a channel of arbitrary shape. Eq.(44) is also referred to as “backwater equation.”

Here, the equation for the gradually varied flow is derived using the energy approach. However, the same equation can be obtained using the momentum approach, which may provide a better insight of the weakness of the governing equation. For example, hydrostatic pressure distribution, which is extremely critical in practice, cannot be seen in the derivation using the energy approach.

6.2 Classifications of the Gradually Varied Flow

The backwater equation given in Eq.(44) is not ready for solution. Since the dependent variable is $h$, $S_e$ and $Fr$ are functions of $h$, the gradually varied flow equation can be re-written as

$$\frac{dy}{dx} = S_0 \frac{1-\frac{y_n}{y}}{1-\left(\frac{y_c}{y}\right)^3}$$  (45)

if the Manning formula is used, and

$$\frac{dy}{dx} = S_0 \frac{1-\frac{y_n}{y}}{1-\left(\frac{y_c}{y}\right)^3}$$  (46)

if the Chezy formula is used.

6.3 Methods of Computation

Computation of gradually varied flows is important in hydraulic engineering practice. In general, the methods of computing the gradually varied flow can be grouped as

- Direct integration method
The direct integration method includes Bress method and Chow method. The direct step method and the standard step method belong to the step method. Since the backwater equation is nonlinear, such method as the Newton-Raphson method can be used. This approach is called numerical method.
Figure 4.10 Various types of non-uniform flows

Figure 11.20 Various types of nonuniform flow with flow from left to right.
7. Unsteady Flows

The continuity and momentum equations for unsteady flows are, respectively, given by

\[
\frac{\partial y}{\partial t} + V \frac{\partial y}{\partial x} + y \frac{\partial V}{\partial x} = 0 \tag{47a}
\]

\[
\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} + g \frac{\partial y}{\partial x} - g (S_o - S_e) = 0 \tag{47b}
\]

This set of equations are called Saint Venant equations or the full-dynamic equations. The momentum equation consists of the local acceleration (the 1st term), the convective acceleration (the second term), the pressure force term (the third term), the gravity force term (the fourth term), and the friction force term (the sixth term).

The set of equations, the continuity and momentum equations, are classified into a second-order hyperbolic type of PDEs. Two characteristics exist, and they propagate in the upstream and downstream directions for the subcritical flow. This feature of PDE enables the method of characteristics to be most accurate and feasible for the solution of the PDE. However, the finite difference method based on the implicit Preissmann scheme became widely used, as indicated in Chow et al. (1988).

As stated, the set of equations is called “the full-dynamic equation” since it includes all terms needed for unsteady flows. This is apparent when compared with such simplified approaches as kinematic wave or diffusion wave approximations. The kinematic wave model only includes the gravity force and friction force terms, and the diffusion wave model includes the pressure force term together with the kinematic wave model.
Example 4.6

Starting with the momentum equation for unsteady flows, show that, for a discharge, stages are different for rising and falling limbs of the flood.

If the both acceleration terms are ignored in the momentum equation, one has

\[ g \frac{\partial y}{\partial x} - g\left(S_0 - S_f\right) = 0 \]

which can be written as

\[ Q = K \sqrt{S_0 - \frac{\partial y}{\partial x}} \]

where \( \frac{\partial y}{\partial x} \) is positive and negative on the falling and rising stages of a flood, respectively.

For the uniform flow, \( Q_0 = K \sqrt{S_0} \)

Example 4.7

Show that the momentum equation for the unsteady flow becomes the backwater equation if \( \frac{\partial V}{\partial t} = 0 \).

If \( \frac{\partial V}{\partial t} = 0 \), then the momentum equation for the unsteady flow becomes

\[ V \frac{dV}{dx} + g \frac{dy}{dx} = g\left(S_0 - S_c\right) \]

where
\[
\nu \frac{dV}{dx} = \frac{1}{2g} \frac{dV^2}{dx} = -\frac{V^2}{gy} \frac{dy}{dx} = -Fr^2 \frac{dy}{dx}
\]

Therefore, we have

\[
\frac{dy}{dx} = \frac{S_0 - S_e}{1 - Fr^2}
\]

References


Problems

1. Normal Depth

Determine the normal depth in a trapezoidal channel carrying a flow of 110 cms. The channel bottom slope is 0.0001, and Manning’s $n = 0.013$, channel bottom width is 20 m, and the side slopes of the channel are 2 horizontal to 1 vertical.

2. Gradually Varied Flow Profile (Julien, p.49)

Consider steady flow in the following impervious rigid boundary channel. The discharge per unit width $q$ is 3.72 m$^2$/s and the water depth at the downstream dam site is 10 m. Assume that the channel width remains large and constant regardless of flow depth, and $f = 0.03$. Determine the distribution of the following variables along the 25 km reach of the channel.

(a) flow depth

(b) mean flow velocity

(c) bed shear stress