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A finite element algorithm for Exner's equation for numerical simulations of 2D morphological change in open-channels

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ABSTRACT: Recently 2D numerical models have been proposed to simulate numerically the morphological change in open-channels. In general, the 2D numerical model for such purpose is comprised of three parts, namely flow, sediment transport, and morphology parts. In the present study, for the flow analysis, the shallow water equations are solved using 2D characteristic dissipative-Galerkin method. In order to update the morphological change, a similar finite element algorithm is proposed for the solution of Exner's equation. The proposed algorithm estimates the morphological change based on sediment loads at Gauss points within the element. On the other hand, the conventional method uses the value at a node, which may result in non-unique values due to the discontinuity of their derivatives. The model is applied to two problems: bed aggradation due to excessive sediment supply at the upstream and propagation of a hump on the bed without sediment supply at the upstream. Appropriate weighting of finite element scheme for the numerical solution of Exner's equation is also investigated. The proposed model is a decoupled model in a sense that the bed elevation does not change simultaneously with the flow during each computational time step, and it is restricted to the case with uniform sediment, neglecting armoring or grain sorting effects.

1 INTRODUCTION

Most 2D numerical models for the simulation of the bed elevation change used the finite difference method. The finite volume method began to be used in 2000s because of its excellent mass conservation property. It is well known that the finite element method provides more flexibility in handling spatial domain than FDM or FVM. Nevertheless, the finite element model has not been favored in the numerical simulations of morphological change compared with the finite difference methods or finite volume methods. Recently, Vasquez et al. (2008) presented the finite element model using triangular mesh for bed elevation change in meandering rivers. However, the detailed finite element algorithm for bed elevation change has not been proposed.

In this study, a finite element model for the flow and bed elevation change is proposed. The shallow water equations and the Exner's equation are solved by the finite element method. The shallow water equations are solved by 2D Characteristic Dissipative-Galerkin (CDG) scheme, which belong to the family of Streamline-Upwind / Petrov-Galerkin (SU/PG) schemea. A new finite element algorithm for the Exner's equation is also introduced, and the new algorithm estimates the equilibrium sediment load not at a node but within an element. In addition, numerical experiments are carried out to find appropriate weight of Exner's equation. For validation, the developed model is applied to straight channel data for bed aggradation due to sediment overloading (Soni et al., 1980) and the propagation of a hump without sediment supply at the upstream. The numerical model developed in the present study is based upon the decoupled modeling approach assuming that the interaction between flow and bed is ignorable during the computational time step. Also, the model is restricted to beds of uniform sediment without armoring or grain sorting effects.

2 NUMERICAL METHODS

2.1 *Flow equations*

For the flow analysis, the following 2D shallow water equations with the effective stress terms are adopted:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{U}}{\partial y} + \frac{\partial \mathbf{D_x}}{\partial x} + \frac{\partial \mathbf{D_y}}{\partial y} + \mathbf{F} = \mathbf{0} \tag{1}$$

where

$$\mathbf{U}^T = \begin{pmatrix} h & p & q \end{pmatrix} \tag{2}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ gh - \dfrac{p^2}{h^2} & 2\dfrac{p}{h} & 0 \\ -\dfrac{pq}{h^2} & \dfrac{q}{h} & \dfrac{p}{h} \end{bmatrix} \tag{3}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ -\dfrac{pq}{h^2} & \dfrac{q}{h} & \dfrac{p}{h} \\ gh - \dfrac{q^2}{h^2} & 0 & 2\dfrac{q}{h} \end{bmatrix} \tag{4}$$

$$\mathbf{D_x} = \begin{bmatrix} 0 \\ -2\nu_t \dfrac{\partial p}{\partial x} \\ -\nu_t \left(\dfrac{\partial p}{\partial y} + \dfrac{\partial q}{\partial x} \right) \end{bmatrix} \tag{5}$$

$$
\mathbf{D_y} = \begin{bmatrix} 0 \\ -\nu_t\left(\dfrac{\partial p}{\partial y}+\dfrac{\partial q}{\partial x}\right) \\ -2\nu_t \dfrac{\partial q}{\partial y} \end{bmatrix} \tag{6}
$$

$$
\mathbf{F} = \begin{bmatrix} 0 \\ gh\dfrac{\partial z_b}{\partial x}+\dfrac{gn^2}{h^{7/3}}p\sqrt{p^2+q^2} \\ gh\dfrac{\partial z_b}{\partial y}+\dfrac{gn^2}{h^{7/3}}q\sqrt{p^2+q^2} \end{bmatrix} \tag{7}
$$

where h is flow depth, p and q are discharge per unit width in x- and y-directions, respectively, g is gravitational acceleration, z_b is bed elevation measured from a certain datum, n is Manning's roughness coefficient, and ν_t is turbulent viscosity. Herein, the following parabolic eddy viscosity model is used:

$$
\nu_t = \frac{\kappa}{6}U_* h \tag{8}
$$

where U_* is the shear velocity, and κ is von Kármán constant (= 0.4).

To solve the shallow water equations numerically, the finite element method is used. The weighted residual equation of the shallow water equations takes the form such as

$$
\int_\Omega N^*\left(\frac{\partial \mathbf{U}}{\partial t}+\mathbf{A}\frac{\partial \mathbf{U}}{\partial x}+\mathbf{B}\frac{\partial \mathbf{U}}{\partial y}+\frac{\partial \mathbf{D_x}}{\partial x}+\frac{\partial \mathbf{D_y}}{\partial y}+\mathbf{F}\right)d\Omega = 0 \tag{9}
$$

where N^* denotes the weighting function. The various finite element schemes have been developed with each unique format. In this study, the Petrov-Galerkin scheme is employed as following:

$$
N_i^* = N_i + \omega\Delta x\frac{\partial N_i}{\partial x}\mathbf{W_x} + \omega\Delta y\frac{\partial N_i}{\partial y}\mathbf{W_y} \tag{10}
$$

where N_i is basis or shape function for the i-th node, N_i^* is weighting function for the i-th node, ω is weighting coefficient, $\mathbf{W_x}$ and $\mathbf{W_y}$ are weighting matrices in the x- and y-directions, respectively, and Δx and Δy are characteristic element lengths in the x- and y- directions, which are estimated by means of Katopodes (1984). In this study, following weighting matrices suggested by Ghanem (1995) are used.

$$
\mathbf{W_x} = \frac{\mathbf{A}}{\sqrt{\mathbf{A}^2+\mathbf{B}^2}}, \quad \mathbf{W_y} = \frac{\mathbf{B}}{\sqrt{\mathbf{A}^2+\mathbf{B}^2}} \tag{11}
$$

where **A** and **B** are advection matrix defined by Eqs. (3) and (4), respectively. For each element, Eq. (9) results nonlinear equations, which are solved by using the Newton-Raphson method and unsymmetrical frontal algorithm proposed by Hood (1976).

2.2 *Exner's equation*

In order to update the bed elevation at each time step, the following Exner's equation is solved:

$$\left(1-p'\right)\frac{\partial z_b}{\partial t}+\frac{\partial q_{tx}}{\partial x}+\frac{\partial q_{ty}}{\partial y}=0 \tag{12}$$

where p' is porosity, and q_{tx} and q_{ty} are the *x*- and *y*-components of total sediment load per unit width, which are, respectively, expressed as

$$q_{tx}=q_t\cos\Phi;\quad q_{ty}=q_t\cos\Phi \tag{13}$$

where Φ is the direction of sediment transport and q_t is the total sediment load per unit width. In the present study, the following formula is used for the total sediment load:

$$q_t=aV^b \tag{14}$$

which was given in Soni et al. (1980). In Eq. (14), V is depth-averaged flow velocity, a and b are constants of values 0.00145 and 5.0, respectively, obtained in Soni et al.'s (1980) experiment.

The weighted residual equation of the Exner's equation is given by

$$\int_\Omega N^*\left[\frac{\partial z_b}{\partial t}+\frac{1}{1-p'}\left(\frac{\partial q_{tx}}{\partial x}+\frac{\partial q_{ty}}{\partial y}\right)\right]d\Omega=0 \tag{15}$$

By using the Green's Theorem in Eq. (15), the following equation in the matrix form can be obtained as proposed by Kim and Choi (2008):

$$\mathbf{A\Delta z_b}=\frac{\Delta t}{1-p'}\left(\mathbf{D}-\mathbf{F}\right) \tag{16}$$

$$A_{ij}=\int_{\Omega^e}\left(N_i^*N_j\right)d\Omega^e \tag{17}$$

$$D_{ij}=\int_{\Omega^e}\left(\frac{\partial N_i^*}{\partial x}q_{tx}+\frac{\partial N_i^*}{\partial y}q_{ty}\right)d\Omega^e \tag{18}$$

$$F_i=\int_{\Gamma^e}\left[N_i^*\left(n_x q_{tx}+n_y q_{ty}\right)\right]d\Gamma^e \tag{19}$$

where Γ^e means the boundary of an element. The conventional method estimates spatial variation of mean velocity and bed topography at a node, which may result in non-unique values due to the discontinuity of their derivatives. However, the proposed algorithm estimates the change of such variables at Gauss points within the element.

In Eq. (15), N^* is the weighting function as in Eq. (9). Unlike the finite scheme for the flow equation, a particular scheme for the Exner's equation has not been proposed nor discussed. Therefore, Bubnov-Galerkin scheme in which the weighting function is the same as the basis function has been just used. In this study, weighting function for the Exner's equation is proposed and applied. A weighting function similar to the one used in PG scheme for the flow equation is used.

$$N_i^* = N_i + \omega \Delta x \frac{q_{tx}}{q_t} \frac{\partial N_i}{\partial x} + \omega \Delta y \frac{q_{ty}}{q_t} \frac{\partial N_i}{\partial y} \qquad (20)$$

If the weighting coefficient, ω, is zero, the weighting function by Eq. (20) results in BG scheme, if positive, upwind weighting scheme, and if negative, downwind weighting scheme, respectively.

3 APPLICATIONS

3.1 *Bed aggradation due to overloaded sediment input*

First, the proposed model is applied to Soni et al.'s (1980) experiment, in which they carried out bed aggradation test due to overloaded sediment supply. The experiment was conducted in a 0.2 m wide, 0.5 m deep, and 30 m long straight tilting flume. Bed materials and supplied sediment particles were uniform sands with a median diameter of 0.32 mm. After the set up of a uniform flow condition for the given discharge and slope, the sediment supply rate was increased to a predetermined value by continuously feeding excess sediment at the upstream end of the flume. The bed and water surface profiles were recorded at time intervals varying from 10 - 20 minutes. The measured profiles were averaged because of the presence of ripples and dunes on the bed.

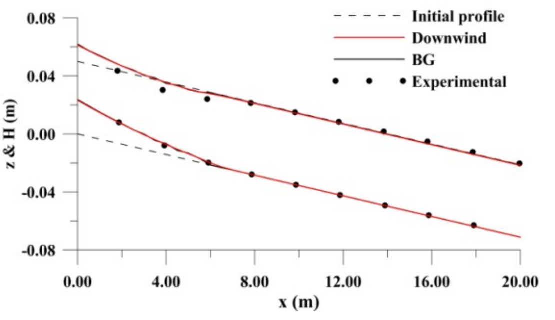
One of the cases in Soni et al.'s (1980) experiments is selected for the numerical simulation. Initial bed slope, water discharge per width, and water depth were 0.00356, 0.02 m^2/s, and 0.05 m, respectively. The amount of sediment supply at the upstream end of the flume is four times larger than the equilibrium sediment load rate. In this application, both BG and downwind schemes are used for the Exner's equation. Figure 1 shows the results of the computed bed and water surface elevation profiles at various times. Due to the overloaded sediment supply at the upstream, bed elevation increases and the range of bed elevation change is extended toward downstream in time. Simulated results agree well with experimental data by Soni et al. (1980), although water surface profiles are slightly higher than measured data.

3.2 *Propagation of hump on the bed*

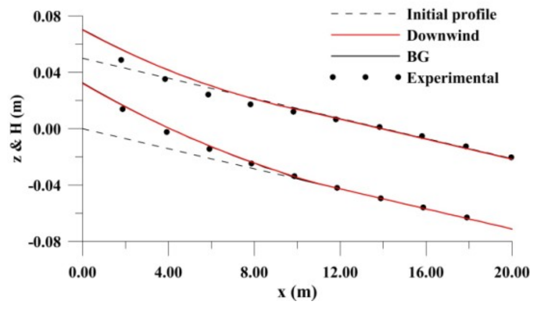
In the present section, the propagation of the hump on the bed is numerically simulated. The flow condition in Soni et al.'s (1980) experiment is imposed with a triangular hump as shown in Figure 2. It is expected that the hump on the bed propagates in the downstream direction with diffusion and equilibrium state reaches.

Figure 3 shows bed elevation profiles simulated by BG scheme. It appears that the hump on the bed spreads with time, reaching equilibrium afterwards. However, it is observed that the disturbance of the bed elevation propagates not only in the downstream but also upstream direction. The propagation towards upstream direction seems to be caused by numerical oscillations.

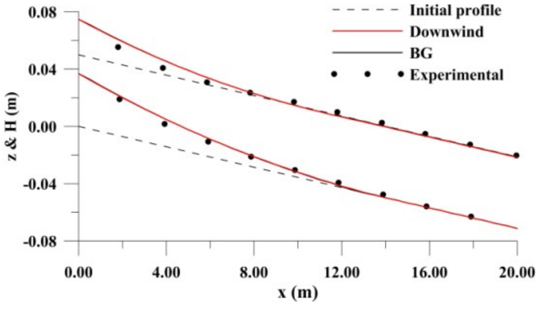
Finally, the downwind or forward weighting scheme with negative weighting coefficient in Eq. (20) is applied. Figure 4 shows simulated bed elevation profiles at various times. It can be seen that the hump propagates mainly in the downstream direction and numerical oscillations are noticeably and rapidly diminished with time. This is comparable with the result of BG scheme in Figure 3.



(a) at 15 min.



(b) at 30 min.



(c) at 40 min.

Figure 1 Simulated bed and water surface profiles

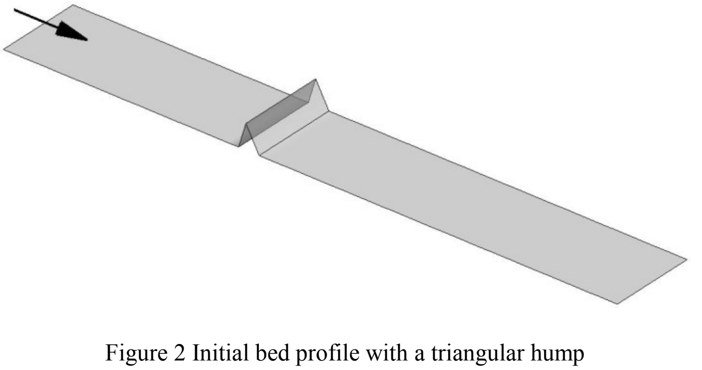
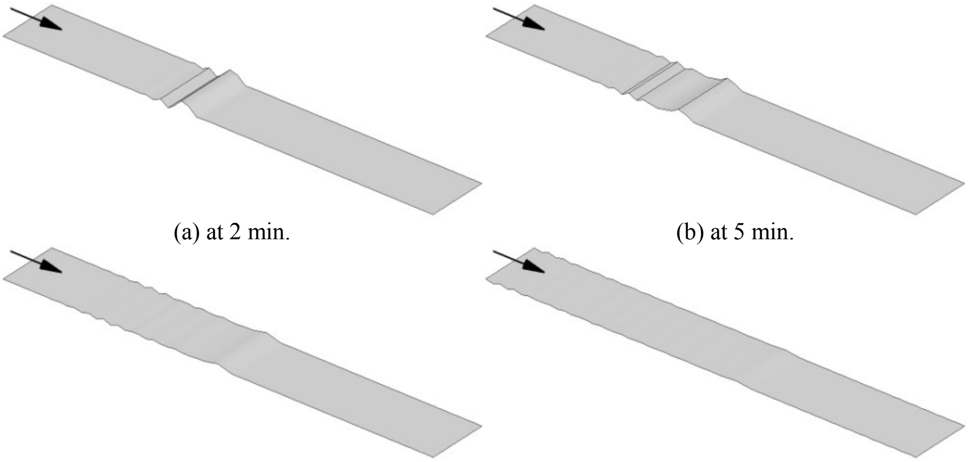


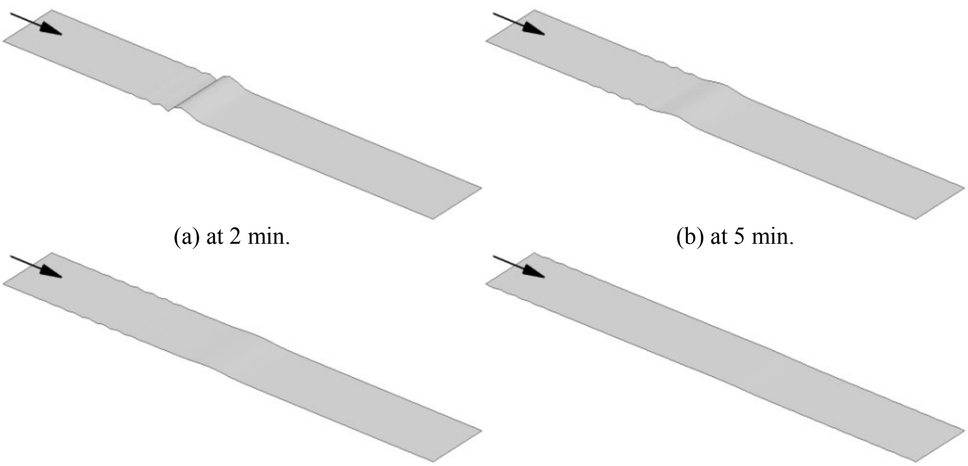
Figure 2 Initial bed profile with a triangular hump



(a) at 2 min. (b) at 5 min.

(c) at 10 min. (d) at 20 min.

Figure 3 Bed elevation change simulated by BG scheme



(a) at 2 min. (b) at 5 min.

(c) at 10 min. (d) at 20 min.

Figure 4 Bed elevation change simulated by downwind weighting scheme

4 CONCLUSIONS

This study presents a finite element model for numerical simulations of 2D morphological change of open-channels. The model computes the flow and morphological change by solving the shallow water equations and Exner's equation, respectively. The 2D characteristic dissipative-Galerkin method is applied to solve the flow equations, and a new algorithm is presented for the solution of Exner's equation. The new algorithm estimates a unique value of the sediment load within the element, which was not possible with conventional methods.

The model was applied to a bed aggradation problem due to sediment over-load at the upstream, Soni et al.'s experiment. It was found that the model reproduce well bed elevation change due to over-loaded sediment supply at the upstream.

Then, the model was applied to the propagation of a hump on the bed. Both BG and downwind schemes simulates well the hump propagation. However, numerical oscillations appear on the bed elevation profile and propagate in the upstream direction when the upwind scheme was used.

5 ACKNOWLEDGEMENTS

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